4.1. Riemannian curvature tensor with constant sectional curvature. Let (M, g) be a Riemannian manifold with constant sectional curvature $\sec(E) = \kappa \in \mathbb{R}$ for all $E \in G_2(TM)$. Show that

$$R(X, Y)W = \kappa \left(g(Y, W)X - g(X, W)Y \right).$$

Solution. Since the sectional curvature is constant, we have

$$R(X, Y, X, Y) = \kappa \left(g(X, X)g(Y, Y) - g(X, Y)g(X, Y) \right)$$

for all $X, Y \in \Gamma(TM)$. Consider now the $(0, 4)$-tensor T given by

$$T(V, W, X, Y) := \kappa \left(g(V, X)g(Y, W) - g(V, Y)g(X, W) \right).$$

Then the $(0, 4)$-tensor $S := R - T$ has the following symmetry properties:

1. $S(V, W, X, Y) = -S(V, W, Y, X)$,
2. $S(V, W, X, Y) + S(V, Y, W, X) + S(V, X, Y, W) = 0$,
3. $S(V, W, X, Y) = S(X, Y, V, W)$,
4. $S(X, Y, X, Y) = 0$.

The first three properties hold for R and T, the last we have already observed above. Our goal is now to show that $S \equiv 0$.

For all $A, B, C, D \in \Gamma(TM)$, we have

$$\begin{aligned} 0 &= S(A, B + D, A, B + D) \\ &= S(A, B, A, B) + S(A, B, A, D) + S(A, D, A, B) + S(A, D, A, D) \\ &= 2S(A, B, A, D) \end{aligned}$$

and

$$\begin{aligned} 0 &= S(A + C, B, A + C, D) \\ &= S(A, B, A, D) + S(A, B, C, D) + S(C, B, A, D) + S(C, B, C, D) \\ &= S(A, B, C, D) + S(A, D, C, B). \end{aligned}$$

Finally, we get

$$\begin{aligned} 3S(V, W, X, Y) &= S(V, W, X, Y) - S(V, Y, X, W) - S(V, W, Y, X) \\ &= S(V, W, X, Y) + S(V, Y, W, X) + S(V, X, Y, W) = 0, \end{aligned}$$

for all $V, W, X, Y \in \Gamma(TM)$. □

4.2. Ricci curvature. Let (M, g) be a 3-dimensional Riemannian manifold. Show the following:

1. The Ricci curvature ric uniquely determines the Riemannian curvature tensor R;
2. If M is an Einstein manifold, then the sectional curvature sec is constant.

Solution. 1. In the following, let e_1, e_2, e_3 be an orthonormal basis of TM_p. First, note that $R_{iijk} = R_{jkii} = 0$ by the symmetry properties of R.

We denote the components of ric by R_{ij}. Then, for $\{i, j, k\} = \{1, 2, 3\}$, we have

$$\begin{aligned} R_{ii} &= R_{iiii} + R_{jiji} + R_{kiki} = R_{ijij} + R_{ikik}, \\ R_{ij} &= R_{iiij} + R_{jijj} + R_{kikj} = R_{ikjk} \end{aligned}$$

and therefore, we get

$$\begin{aligned} 2R_{ijij} &= R_{ii} + R_{jj} - R_{kk}, \\ R_{ikjk} &= R_{ij}. \end{aligned}$$

Observe now, that we can compute all other components of R by symmetry properties. Hence R is uniquely determined by ric.

2. We assume that ric $= kg$. Let e_1, e_2 be a orthonormal basis of $E \subset TM_p$ and choose e_3 such that e_1, e_2, e_3 is an orthonormal basis of TM_p. Then we have

$$2 \sec_p(E) = 2R_{1212} = R_{11} + R_{22} - R_{33} = k + k - k = k$$

and hence $\sec_p(E) = \frac{k}{2}$. □

4.3. Divergence and Laplacian. Let (M, g) be a Riemannian manifold with Levi-Civita connection D. The *divergence* $\operatorname{div}(Y)$ of a vector field $Y \in \Gamma(M)$ is the contraction of the $(1,1)$-tensor field $DY : X \mapsto D_X Y$ and the *Laplacian* $\Delta : C^\infty(M) \to C^\infty(M)$ is defined by $\Delta f := \operatorname{div}(\operatorname{grad} f)$. Show that:

1. $\operatorname{div}(fY) = Y(f) + f \operatorname{div} Y$;
2. $\Delta(fg) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g \rangle$;
3. Compute Δf in local coordinates.

Solution. 1. Let $p \in M$ and let $e_1, \dots, e_n$ be a orthonormal basis of TM_p. Then we have

$$\begin{aligned}
\operatorname{div}_p(fY) &= \sum_{i=1}^{n} \langle D_{e_i}(fY), e_i \rangle \\
&= \sum_{i=1}^{n} \langle e_i(f) Y_p + f(p) D_{e_i} Y, e_i \rangle \\
&= \sum_{i=1}^{n} e_i(f) \langle Y_p, e_i \rangle + \sum_{i=1}^{n} f(p) \langle D_{e_i} Y, e_i \rangle \\
&= Y_p(f) + f(p) \operatorname{div}_p(Y) \\
&= (Y(f) + f \operatorname{div}(Y))(p)
\end{aligned}$$

and hence $\operatorname{div}(fY) = Y(f) + f \operatorname{div} Y$.

2. First, recall the definition of $\operatorname{grad} f$ from Serie 2, i.e. $X(f) = \langle \operatorname{grad} f, X \rangle$ and note that

$$\langle \operatorname{grad}(fg), X \rangle = X(fg) = X(f)g + fX(g) = \langle \operatorname{grad}(f)g + f \operatorname{grad}(g), X \rangle,$$

for all $X \in \Gamma(M)$ and thus $\operatorname{grad}(fg) = \operatorname{grad}(f)g + f \operatorname{grad}(g)$.

Therefore, we get

$$\begin{aligned}
\Delta(fg) &= \operatorname{div}(\operatorname{grad}(fg)) \\
&= \operatorname{div}(\operatorname{grad}(f)g + f \operatorname{grad}(g)) \\
&= \operatorname{div}(\operatorname{grad} f)g + \operatorname{grad}(f)(g) + f \operatorname{div}(\operatorname{grad}(g)) + \operatorname{grad}(g)(f) \\
&= \Delta(f)g + f\Delta(g) + 2\langle \operatorname{grad} f, \operatorname{grad} g \rangle.
\end{aligned}$$

3. In the following, we use Einstein notation. With the notion of Exercise sheet 2, we already know that

$$\begin{aligned}
\operatorname{grad} f &= g^{ik} f_i A_k, \\
D_{A_i} Y &= (A_i(Y^k) + Y^j \Gamma_{ij}^k) A_k.
\end{aligned}$$

Therefore, we have

$$\operatorname{div}(Y) = A_k(Y^k) + Y^j \Gamma_{kj}^k.$$

Hence, we get

$$\begin{aligned}
\Delta f &= \operatorname{div}(\operatorname{grad} f) \\
&= A_k(g^{ik} f_i) + g^{ij} f_i \Gamma_{kj}^k.
\end{aligned}$$

Recall Jacobi's formula:

$$\operatorname{tr}(B^{-1}dB) = \frac{1}{\det B} d \det B,$$

for any invertible matrix B. Using the formula with $B = (g_{ij})$ and denoting $G := \det(g_{ij})$, we have:

$$\Gamma^k_{kj} = \frac{1}{2} g^{kl} A_j(g_{kl}) = \frac{1}{2G} A_j(G) = \frac{1}{\sqrt{G}} A_j(\sqrt{G})$$

and therefore

$$\begin{aligned}\Delta f &= A_j(g^{ij} f_i) + g^{ij} f_i \frac{1}{\sqrt{G}} A_j(\sqrt{G}) \\ &= \frac{1}{\sqrt{G}} A_j \left(\sqrt{G} g^{ij} f_i \right).\end{aligned}$$

□